

Unimodular Gravity with Pseudo-scale Invariance

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Abstract: We consider a model of gravity and matter fields which is invariant only under unimodular general coordinate transformations (GCT). The determinant of the metric is treated as a separate field which transforms as a scalar under unimodular GCT. Furthermore we also demand that the theory obeys pseudo-scale invariance. We study the implications of the resulting theory. We solve the resulting field equations for a spherically symmetric system in vacuum. We find that the resulting solution contains an additional term in comparison to the standard Schwarzschild solution. We also study the cosmological implications of the model. We find that, in terms of cosmic time, it predicts an accelerated expansion if the energy density is dominated by non-relativistic baryonic matter. Furthermore the model does not admit a cosmological constant, thereby solving its fine tuning problem.

1 Introduction

In a recent paper [1] we have considered some implications of a model which is invariant only under the restricted unimodular general coordinate transformations (GCT) but not the full GCT. The basic idea that the gravitational action may be invariant only under the unimodular GCT was first proposed by Anderson and Finkelstein [2]. It is related to an earlier proposal by Einstein [3]. In this proposal the determinant of the metric, g , is not a dynamical variable. This idea has been pursued in detail in many papers [4–13], which have considered its application to the problem of the cosmological constant and its quantization. Since g is not a dynamical field, a model based on unimodular GCT has the potential to solve the cosmological constant problem. However as explained in [14], the problem is not really solved.

In the present paper we treat the determinant as an independent scalar field since we only demand invariance under unimodular general coordinate transformations (GCT) [10, 14–19]. However we also require that the model is invariant under the pseudo-scale invariance [20]. This restricts the action considerably. We study some implications of the resulting model. In particular we study the cosmological evolution implied by this model as well as its spherically symmetric solution in vacuum.

2 Review of Unimodular Gravity

We require only unimodular general coordinate transformations (GCT), which are defined by,

$$x^\mu \rightarrow x'^\mu \tag{1}$$

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such that

$$\det(\partial x'^{\mu}/\partial x^{\nu}) = 1 \quad (2)$$

It is convenient to split the standard metric as follows [1],

$$g_{\mu\nu} = \chi^2 \bar{g}_{\mu\nu} \quad (3)$$

where the determinant \bar{g} of $\bar{g}_{\mu\nu}$ is assumed to be non-dynamical. Hence we demand that \bar{g} is fixed such that,

$$\bar{g} = \det[\bar{g}_{\mu\nu}] = f(x) \quad (4)$$

where $f(x)$ is some function of the space-time coordinates. The field χ behaves as a scalar field under unimodular GCT. Hence the basic fields of our theory are χ , the metric $\bar{g}_{\mu\nu}$ and matter fields. We denote the connection, the Ricci tensor and the curvature scalar by the symbols $\bar{\Gamma}_{\alpha\beta}^{\mu}$, $\bar{R}_{\mu\nu}$ and \bar{R} respectively. All these quantities are computed by using the metric $\bar{g}_{\mu\nu}$.

3 Unimodular gravity with pseudo-scale invariance

We next present a model of gravity and matter fields which is invariant under unimodular GCT but not the full GCT. As discussed in [1, 10, 14, 15, 19] there is considerable freedom in writing such a model. We impose a further constraint on this model that it should satisfy global pseudo-scale invariance [20]. As explained in Ref. [20] under this transformation, the coordinates x^{μ} do not change. However the metric and the fields undergo suitable transformation. In 4 space-time dimensions, the full metric essentially transforms as

$$g_{\mu\nu} \rightarrow g_{\mu\nu} \Lambda^2 \quad (5)$$

where Λ is a constant parameter. This transformation changes the determinant of the metric. Hence in our case we may express this transformation as,

$$\begin{aligned} \bar{g}_{\mu\nu} &\rightarrow \bar{g}_{\mu\nu} \\ \chi &\rightarrow \chi \Lambda \end{aligned}$$

The matter fields transform as follows,

$$\begin{aligned} \phi &\rightarrow \phi/\Lambda \\ A_{\mu} &\rightarrow A_{\mu} \\ \psi &\rightarrow \psi/\Lambda^{3/2} \end{aligned}$$

where ϕ , A_{μ} and ψ are scalar, vector and spinor fields respectively.

The action invariant under unimodular GCT and pseudo-scale transformations may be written as,

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{\kappa} \bar{R} - \frac{\xi}{\kappa} \bar{g}^{\mu\nu} \partial_{\mu} \ln \chi \partial_{\nu} \ln \chi \right] + S_M \quad (6)$$

where S_M represents the matter part of the action and $\kappa = 16\pi G$. It is useful to compare this action with the standard gravitational action written in terms of $\bar{g}_{\mu\nu}$ and χ [1]. We see that the current action represents a significant modification of the standard Einstein's action. Hence it is likely to give very different predictions and should be carefully examined to see if it agrees with

observations.

We next display the matter action which contains the standard model fields. Before doing that we also need to suitably split the vierbi field to be consistent with Eq. 3. We consider the vierbi field e_i^a where a represent the Lorentz index. The full metric

$$g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b \quad (7)$$

We may split

$$e_\mu^a = \chi \bar{e}_\mu^a \quad (8)$$

Here we have defined \bar{e}_μ^a such that it has determinant equal to $\sqrt{-g}$. We may now write S_M as

$$\begin{aligned} S_M = & \int d^4x \sqrt{-g} \left[\chi^2 \bar{g}^{\mu\nu} (D_\mu \mathcal{H})^\dagger (D_\nu \mathcal{H}) - \frac{1}{4} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} (\mathcal{A}_{\mu\alpha}^i \mathcal{A}_{\nu\beta}^i + \mathcal{B}_{\mu\alpha} \mathcal{B}_{\nu\beta} + \mathcal{G}_{\mu\alpha}^j \mathcal{G}_{\nu\beta}^j) \right. \\ & \left. + m^2 \chi^2 \mathcal{H}^\dagger \mathcal{H} - \lambda \chi^4 (\mathcal{H}^\dagger \mathcal{H})^2 \right] + \mathcal{S}_{\text{fermions}}, \end{aligned} \quad (9)$$

where \mathcal{H} is the Higgs doublet, $\mathcal{G}_{\mu\nu}^j$, $\mathcal{A}_{\mu\nu}^i$, and $\mathcal{B}_{\mu\nu}$ represent the field strength tensors for the $SU(3)$, $SU(2)$ and $U(1)$ vector fields respectively. The superscripts i and j on $\mathcal{A}_{\mu\nu}^i$ and $\mathcal{G}_{\mu\nu}^j$ represent the $SU(2)$ and $SU(3)$ indices respectively. We are implicitly summing over these indices. The fermion action may be expressed as,

$$\begin{aligned} \mathcal{S}_{\text{fermions}} = & \int d^4x \bar{e} (\chi^3 \bar{\psi}_L i \gamma^\mu \mathcal{D}_\mu \psi_L + \chi^3 \bar{\psi}_R i \gamma^\mu \mathcal{D}_\mu \psi_R) \\ & - \int d^4x \bar{e} (g_Y \chi^4 \bar{\psi}_L \mathcal{H} \psi_R + h.c.), \end{aligned} \quad (10)$$

where $e = \sqrt{-g}$, $\gamma^\mu = \bar{e}_a^\mu \gamma^a$ and a, b are Lorentz indices. Here ψ_L is an $SU(2)$ doublet, ψ_R a singlet and g_Y represents a Yukawa coupling. For simplicity we have displayed only one Yukawa coupling term. Furthermore we have displayed the action only for a single family.

The covariant derivative acting on the fermion field is defined by

$$\mathcal{D}_\mu \psi_{L,R} = \left(\tilde{D}_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} \right) \psi_{L,R}, \quad (11)$$

where $\tilde{D}_\mu \psi_L = \partial_\mu - ig \mathbf{T} \cdot \mathbf{A}_\mu - ig'(Y_f^L/2) B_\mu$, $\tilde{D}_\mu \psi_R = \partial_\mu - ig'(Y_f^R/2) B_\mu$, $\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]$ and ω_μ^{ab} is the spin connection. In Eq. 11, A_μ is the $SU(2)$ field, B_μ the $U(1)$ field, \mathbf{T} represents the $SU(2)$ generators and Y_f the $U(1)$ hypercharges. Here we have not explicitly displayed the color interactions for quarks, which can be easily added.

We point out that the matter action is essentially the same as in the case of generally covariant standard model, except for a crucial difference. The mass term of the Higgs field is different. If we were to preserve general covariance then this term would be multiplied by an additional factor of χ^2 . This term breaks GCT but preserves pseudo-scale invariance. We may also point out that the pseudo-scale invariance does not allow a cosmological constant term in the action. As we shall see this invariance is preserved exactly in our theory. It is not broken spontaneously. This symmetry allows a mass term in the action, provided we break the full GCT. Hence we can generate masses of all particles without breaking this symmetry. Furthermore, this symmetry can also be preserved in the full quantum theory [16]. When we quantize the theory the symmetry may be broken due to an anomaly. The anomalies arise due to the fact that the process of regulating the theory does not

preserve some of the symmetries of the classical action. For example, the standard scale invariance is anomalous since the regulated action necessarily contains a scale. This has been shown explicitly in the context of dimensional regularization in Ref. [22]. The pseudo-scale invariance, however, may be preserved in the dimensionally regulated action by introducing appropriate fractional powers of χ , or equivalently the determinant of the full metric, $g_{\mu\nu}$, in some of the terms in the action [16]. This follows a similar suggestion made earlier in the case of scale symmetry [23]. This procedure has also been employed in the case of unimodular gravity with scale invariance [24, 25] as well as for computing the cosmological constant in a scale invariant model [26]. In our model this symmetry implies that cosmological constant remains zero in the full theory and hence solves its fine tuning problem. The basic issue now is whether the theory agrees with observations. We partially address this issue in the present paper.

We point out that since the theory has unimodular general coordinate invariance, the energy momentum tensor of the matter fields satisfies a conservation law,

$$(\mathcal{T}_\nu^\mu)_{;\mu} = 0 \quad (12)$$

provided the contribution due to the scalar field χ is also included. The symbol \mathcal{T}_ν^μ is used in Eq. 12 to indicate that the contribution due to the field χ is also included. In Eq. 12, the covariant derivative as well as all raising and lowering of indices is done using the metric $\bar{g}_{\mu\nu}$.

4 Spherically Symmetric Solution in Vacuum

We next determine whether our model reproduces the standard spherically symmetric Schwarzschild solution. Imposing the unimodular constraint on the metric $\bar{g}_{\mu\nu}$ we can write it as

$$\bar{g}_{\mu\nu} = \text{diag} \left[\frac{1}{A(r)}, -A(r), -r^2, -r^2 \sin^2 \theta \right] \quad (13)$$

The full metric is given by Eq. 3 where $\chi = \chi(r)$. The determinant $\det[\bar{g}_{\mu\nu}]$ is equal to the determinant of the Lorentz metric. The curvature tensor satisfies the following equation in vacuum,

$$-\left[\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}\right] + \xi \left[\partial_\mu \ln \chi \partial_\nu \ln \chi - \frac{1}{4}\bar{g}_{\mu\nu} \partial^\lambda \ln \chi \partial_\lambda \ln \chi \right] = 0 \quad (14)$$

We have

$$\frac{\bar{R}_{rr}}{A} + A\bar{R}_{tt} = 0 \quad (15)$$

which gives,

$$\partial_r \ln \chi = 0 \quad (16)$$

Hence χ is a constant. This implies that

$$\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R} = 0 \quad (17)$$

This is as far as we can go. Now the Schwarzschild solution indeed solves this equation. However it is not unique. We cannot set $\bar{R} = 0$. We find,

$$\bar{R}'_{\theta\theta} - \frac{2}{r}\bar{R}_{\theta\theta} = 0 \quad (18)$$

where the prime refers to derivative with respect to r . Hence

$$\bar{R}_{\theta\theta} = C_2 r^2 \quad (19)$$

where C_2 is a constant. This implies

$$rB' + B = 1 + C_2 r^2 \quad (20)$$

where $B = 1/A$. If we set $C_2 = 0$ we get the standard Schwarzschild solution. This also applies approximately for small r . However for large r the solution gets modified. We find,

$$B(r) = \frac{1}{A(r)} = 1 + \frac{C_3}{r} + \frac{r^2}{3} C_2 \quad (21)$$

where C_3 is another constant. By using the relationship $B = 1 + 2\phi$, where ϕ is the gravitational potential, we can relate C_3 to the mass M of the source in the usual manner. We have, neglecting the term proportional to C_2 for small r ,

$$C_3 = -2GM \quad (22)$$

where G is the gravitational constant. Hence we find the gravitational potential,

$$\phi = -\frac{GM}{r} + \frac{C_2}{6} r^2 \quad (23)$$

We find that the potential deviates from the standard Newtonian potential at large distances. It is clearly of interest to see if this can explain the galactic rotation curves. By relating the gravitational force due to this potential on a test mass in circular motion with speed v at distance r from the source, we find that, at large distances,

$$v \approx \sqrt{\frac{C_2}{3}} r \quad (24)$$

Hence the rotational speed increases linearly with r . The observed rotational speeds in many cases increase slowly with distance from the center and hence might be consistent with the predicted behaviour. A detailed fit is required which must take into account the modification of our prediction due to visible mass in the galactic halo. This is postponed to future work.

5 Cosmological Implications

We next study the cosmological implications of our unimodular pseudo-scale invariant theory. The generalization of Einstein's equations to the present theory, is given by,

$$-\left[\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}\right] + \xi\left[\partial_\mu \ln \chi \partial_\nu \ln \chi - \frac{1}{4}\bar{g}_{\mu\nu}\partial^\lambda \ln \chi \partial_\lambda \ln \chi\right] = \frac{\kappa}{2}\left[T_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}T^\lambda_\lambda\right] \quad (25)$$

Here $T_{\mu\nu}$ represents all the contributions to this equation obtained by the matter action. We may call this the energy momentum tensor. However we caution the reader that it does not satisfy the usual conservation law. The equation of motion for χ may be written as

$$2\xi\bar{g}^{\mu\nu}(\ln \chi)_{;\mu;\nu} = \kappa\chi T_\chi \quad (26)$$

where T_χ represents the contributions to this equation due to the matter fields.

We shall assume, for simplicity, that

$$\bar{g}_{\mu\nu} = \text{diagonal}[1, -1, -1, -1] \quad (27)$$

Hence the entire dynamics is contained in the field χ . Here we shall first consider the case where the energy density of the universe is dominated by radiation and next where it is dominated by non-relativistic matter. As discussed earlier, vacuum energy or cosmological constant does not contribute in our theory.

5.1 Radiation dominated universe

In the case radiation dominated universe, we find that,

$$T_\chi = 0 \quad (28)$$

We see this easily from the matter action. For the case of a vector field, there is no direct coupling to χ . Hence its contribution to T_χ vanishes trivially. In the case of scalar or spinor field, it vanishes once we impose the condition $P^2 = 0$. Hence in this case we find that,

$$\frac{d^2}{d\eta^2} \ln \chi = 0 \quad (29)$$

Here η is the time coordinate. Note that the time coordinate in our metric essentially corresponds to the conformal time in the standard Big Bang cosmology. This implies that, $d \ln \chi / d\eta = C$, where C is a constant. Hence

$$\chi = \chi_0 e^{C\eta} \quad (30)$$

Here χ_0 is a constant. We may express this result in terms of cosmic time, t , by using,

$$\chi d\eta = dt \quad (31)$$

In terms of cosmic time we find that the universe is expanding at a constant rate. The constant C is fixed by Eq. 25. In this equation $\bar{R}_{\mu\nu} = 0$, $\bar{R} = 0$. We may identify the energy momentum tensor by taking the specific example of a scalar field [1],

$$T_{\mu\nu} = \chi^2 < \partial_\mu \phi \partial_\nu \phi > \quad (32)$$

where the expectation value is taken in an appropriate thermal state corresponding to the temperature of the medium. Furthermore, as discussed in Ref. [1]

$$< \partial_0 \phi \partial_0 \phi > = \chi^2(\eta) \rho \quad (33)$$

where ρ is the energy density of the radiation field. We also find that the trace, $T_\chi^\lambda = 0$. Substituting in Eq. 25, we find

$$\rho = \frac{3\xi C^2}{2\kappa \chi^4} \quad (34)$$

Hence we find the standard result, $\rho \propto 1/\chi^4$. However the time dependence of the scale factor is very different. We find an exponential dependence on conformal time instead of the usual linear dependence in the case of Einstein's gravity [1].

The dependence of the electromagnetic radiation energy density on the scale factor χ may also be obtained directly from the conservation law, Eq. 12. The electromagnetic field does not directly couple to the scalar field χ . Hence we need not include the contribution due to χ while applying Eq. 12. As argued above and in Ref. [1], we identify,

$$T_{\mu\nu} = \chi^4 \text{ diagonal}(\rho, -p, -p, -p) \quad (35)$$

where p denotes the pressure. Setting $\nu = 0$ in Eq. 12, we find,

$$\frac{d(\rho\chi^4)}{d\eta} = 0 \quad (36)$$

which implies that $\rho \propto 1/\chi^4$ in agreement with Eq. 34. We may similarly obtain the evolution of ρ for all fields which do not directly couple to χ .

We next consider the evolution of a scalar field ϕ in the relativistic limit. In this case we need to include the contribution due to χ . We find

$$\mathcal{T}_{\mu\nu} = -\frac{2\xi}{\kappa} \partial_\mu \ln \chi \partial_\nu \ln \chi + T_{\mu\nu}(\phi) \quad (37)$$

where $T_{\mu\nu}(\phi)$ is the contribution due to the scalar field and is given by Eq. 35. Using the conservation law, Eq. 12 with $\nu = 0$, we find

$$\frac{d(\rho\chi^4)}{d\eta} - \frac{2\xi}{\kappa} \frac{d}{d\eta} \left(\frac{d}{d\eta} \ln \chi \right)^2 = 0 \quad (38)$$

This implies that

$$\rho\chi^4 - \frac{2\xi}{\kappa} \left(\frac{d}{d\eta} \ln \chi \right)^2 = \text{constant} \quad (39)$$

Now using Eq. 29, we find that $\rho \propto 1/\chi^4$ in this case also.

5.2 Non-relativistic Matter dominated universe

We first consider a real scalar field ϕ in the non-relativistic limit. The action for the field can be written as

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{1}{2} \chi^2 \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \chi^2 \phi^2 \right], \quad (40)$$

In this case also we find that $T_\chi = 0$, after using the field equation of ϕ . Hence we again find $d \ln \chi / d\eta = C$, where C is a constant, as in the case of radiation dominated universe. Using Eq. 25, we find the energy density, ρ , given by

$$\rho = \frac{2\xi C^2}{\kappa \chi^4} \quad (41)$$

Hence in this case we again find that the energy density falls a $1/\chi^4$, as in the case of radiation. We may understand this surprising result by noticing that in the present case the mass term of the scalar particle has two powers of χ less than that of standard gravity which has full covariance. A Dirac fermion with an explicit mass term in the action also behaves in a similar manner.

We next consider the case of chiral fermions. In this case we do not introduce an explicit mass term in the action, as in the case of the Standard Model of particle physics. The fermion

mass is generated by the Higgs mechanism through the Yukawa interaction terms. In this case, surprisingly, we get a different result. We find a non-zero contribution to T_χ in contrast to the case when we directly introduce a mass term of fermions in the action.

We consider the standard case of visible matter which is composed of spin 1/2 particles. The mass of these particles is generated through the Higgs mechanism by the Yukawa interaction terms. Here we focus on elementary particles such as electrons. We shall discuss the generalization to composite particles such as protons later in this section. The contribution to the equation of motion of χ is given by

$$T_\chi = -3\chi^2 \langle \bar{\psi}_L i \bar{e}_a^\mu \gamma^a \partial_\mu \psi_L + (L \rightarrow R) \rangle + 4g_Y \chi^3 \langle \bar{\psi}_L \mathcal{H} \psi_R + h.c. \rangle \quad (42)$$

The contribution of the Higgs field to T_χ vanishes after using its equation of motion. The Higgs field acquires non-zero vacuum expectation value due to spontaneous symmetry breaking. Hence we find

$$\langle \mathcal{H} \rangle = \mathcal{H}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad (43)$$

where

$$v^2 = \frac{m^2}{\lambda \chi^2} \quad (44)$$

Interestingly we find that v depends on the scale factor and hence keeps decreasing as the universe expands. We may identify the mass of the fermion as,

$$M_\psi = \frac{g_Y m}{\sqrt{2} \lambda} \quad (45)$$

The equation of motion for χ finally gives,

$$2\xi \bar{g}^{\mu\nu} (\ln \chi)_{;\mu;\nu} = \kappa \chi^3 M_\psi \langle \bar{\psi} \psi \rangle = \kappa \chi^4 \rho \quad (46)$$

where ρ is the energy density of the non-relativistic matter. The identification of ρ follows by considering the energy momentum tensor, as in the case of radiation field. We have

$$T_{\mu\nu} = \chi^3 \langle \bar{\psi}_L i \bar{e}_a^\alpha \gamma^a \bar{g}_{\alpha\mu} \partial_\nu \psi_L + (L \rightarrow R) \rangle \quad (47)$$

We are interested in the 00 component of this equation. As in the case of radiation field we have

$$T_{00} = \chi^4 \rho \quad (48)$$

Using Eq. 25, this gives

$$\xi \left(\frac{d}{d\eta} \ln \chi \right)^2 = \frac{\kappa}{2} \chi^4 \rho \quad (49)$$

From Eqs. 49 and 46 we find,

$$\frac{d}{d\eta} \left(\frac{d \ln \chi}{d\eta} \right) = \left(\frac{d \ln \chi}{d\eta} \right)^2 \quad (50)$$

This gives

$$\dot{\chi} = C_1 \chi^2 \quad (51)$$

where the dot denotes derivative with respect to time η and C_1 is a constant. Substituting this

in Eq. 49 we find,

$$\rho = \frac{2\xi C_1^2}{\kappa\chi^2} \quad (52)$$

Hence we find that in this case $\rho \propto 1/\chi^2$, which is different from the $1/\chi^3$ behaviour found in the standard case of Einstein's relativity and the $1/\chi^4$ behaviour found above. We point out that Eq. 52 relates the constant C_1 to the energy density and the scale factor today. Furthermore we notice that Eq. 51 is exactly the same as the equation for the scale factor corresponding to vacuum energy in usual Einstein's gravity, expressed in terms of conformal time. Hence if we make a change of variables from conformal time η to cosmic time t , we find the standard equation for vacuum dominated universe,

$$\frac{d\chi}{dt} = C_1\chi \quad (53)$$

However this behaviour is now being obtained for non-relativistic matter rather than vacuum energy. Hence we find the very interesting result that Standard Model fermions in the non-relativistic limit lead to acceleration which is exactly the same as that obtained by vacuum energy in the case of usual Big Bang cosmology. Hence despite the fact that we have null cosmological constant in our theory we have obtained an exponentially expanding solution.

So far we have discussed the contribution due to non-relativistic elementary fermions, such as electrons. For composite objects such as protons, neutrons, the situation is more complicated. However in this case also the entire mass is generated due to spontaneous or dynamical symmetry breaking. The bare mass of the quarks is generated by the vacuum expectation value of the Higgs field. To a good approximation this is negligible for light quarks. The mass of the proton may be modelled by using the SU(2) linear sigma model which includes the pion and sigma spin 0 fields and the proton, neutron as the fermion doublet [21]. For our purpose the mass is generated essentially in the same manner as that of electrons in the Standard Model. Hence it will behave in the same manner cosmologically. The equations given in this section for elementary fermions are also applicable to composite fermions such as protons and neutrons.

5.3 Evolution of energy density for non-relativistic field

We have already seen that during the phase when the energy density of Standard Model fermions, in the non-relativistic limit, dominates, the energy density decays as $1/\chi^2$. We next ask the question how does the energy density of such fermions decays during the radiation dominated phase. As we saw in section 5.1 the conservation law, Eq. 12, explicitly involves the dynamics of the field χ and hence may not be very useful for this purpose. Hence we try to address this question directly from the equations of motion by assuming a subdominant contribution due to non-relativistic particles. The equations of motion may now be expressed as

$$\begin{aligned} \frac{d^2}{d\eta^2} \ln \chi &= \frac{\kappa}{2\xi} \chi^4 \rho_N \\ \left(\frac{d}{d\eta} \ln \chi \right)^2 &= \frac{\kappa}{2\xi} \chi^4 \left(\rho_N + \frac{4}{3} \rho_R \right) \end{aligned} \quad (54)$$

where ρ_N is the energy density of the non-relativistic field of the type considered in section 5.2 and ρ_R is the energy density of a relativistic field. We assume that $\rho_R \gg \rho_N$. At leading order we neglect ρ_N and the solution is $d \ln \chi / d\eta = C$, where C is a constant, such that

$$C^2 = \frac{2\kappa}{3\xi} \chi^4 \rho_R \quad (55)$$

Including the corrections due to ρ_N we may write

$$\frac{d}{d\eta} \ln \chi = C + \delta(\eta) \quad (56)$$

The field equations imply,

$$\begin{aligned} \frac{d\delta}{d\eta} &= \frac{\kappa}{2\xi} \chi^4 \rho_N \\ \delta &= \frac{\kappa}{4\xi C} \rho_N \chi^4 \end{aligned} \quad (57)$$

Hence we find

$$\frac{d\delta}{d\eta} = 2C\delta \quad (58)$$

which gives

$$\delta \propto e^{2C\eta} \propto \chi^2 \quad (59)$$

Hence we find that $\rho_N \propto 1/\chi^2$ during the phase of radiation domination also.

Before ending this section we summarize the main results. We find that the evolution of the scale factor, both during radiation and matter dominated phase to be different from what is found in the case of standard Einstein's gravity. If the energy density is dominated by non-relativistic baryonic matter we find an accelerated expansion. The energy density of electromagnetic field falls as $1/\chi^4$ during all the phases. During the radiation dominated era, the energy density of all relativistic fields also falls as $1/\chi^4$. Surprisingly, however, the energy density of a real scalar field, as well as dirac fermions with an explicit mass term, in the non-relativistic limit also falls as $1/\chi^4$ in contrast to the result in the standard Big Bang model. We also find that the energy density of Standard Model fermions in the non-relativistic limit falls as $1/\chi^2$ in contrast to the $1/\chi^3$ obtained in the case of Einstein's gravity. In this case the scale factor shows expansion similar to that obtained by vacuum dominated universe in the case of standard Einstein's gravity.

6 Discussion and Conclusions

We have considered a model which obeys only unimodular general coordinate invariance. We impose an additional requirement that the theory also has pseudo-scale invariance. We have shown that the theory admits the standard Schwarzschild solution for a spherically symmetric system in vacuum only as a special case. In general the solution contains an extra term which leads to a modification of the Newtonian potential. We have also considered the cosmological implications of this theory. The theory does not admit a cosmological constant and hence solves its fine tuning problem. We find that, if the energy density of the universe is dominated by the non-relativistic baryonic matter, the scale factor shows accelerated expansion in terms of cosmic time. Hence the theory has the potential to agree with cosmological data without requiring dark energy.

Acknowledgements

We thank Subhadip Mitra, Sukanta Panda, V. Ravishankar and Kandaswamy Subramanian for useful discussions.

References

- [1] P. Jain, P. Karmakar, S. Mitra, S. Panda and N. K. Singh, arXiv:1108.1856.
- [2] J.L. Anderson and D. Finkelstein, Am. J. Phys. **39**, 901 (1971).
- [3] A. Einstein, in *The Principle of Relativity*, edited by A. Sommerfeld (Dover, New York, 1952).
- [4] J. J. van der Bij, H. van Dam and Y. J. Ng, Physica **116A**, 307 (1982).
- [5] M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989).
- [6] W. G. Unruh, Phys. Rev. D **40**, 1048 (1989).
- [7] Y. J. Ng and H. van Dam, J. Math. Phys. **32**, 1337 (1991).
- [8] D. R. Finkelstein, A. A. Galiautdinov and J. E. Baugh, J. Math. Phys. **42**, 340 (2001) [arXiv:gr-qc/0009099].
- [9] A. Zee, in *High Energy Physics: Proceedings of the 20th Annual Orbis Scientiae, 1983*, edited by S. L. Mintz and A. Perlmutter (Plenum, New York).
- [10] W. Buchmuller and N. Dragon, Phys. Lett. **B207**, 292.
- [11] E. Alvarez, JHEP **0503**, 002 (2005) [arXiv:hep-th/0501146].
- [12] E. Alvarez, D. Blas, J. Garriga, E. Verdaguer Nucl. Phys. B **756**, 148 (2006) [arXiv:hep-th/0606019].
- [13] G. F. R. Ellis, H. van Elst, J. Murugan and J. P. Uzan, [arXiv:1008.1196 (gr-qc)].
- [14] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
- [15] E. Alvarez and A. F. Faedo, Phys. Rev. D **76**, 064013 (2007) [arXiv:hep-th/0702184].
- [16] P. Jain, S. Mitra and N. K. Singh, JCAP **0803**, 011 (2008) [arXiv:0801.2041 [astro-ph]].
- [17] E. Alvarez, A. F. Faedo and J. J. Lopez-Villarejo, JCAP **0907**, 002 (2009) [arXiv:0904.3298 (hep-th)].
- [18] E. Alvarez and R. Vidal, Phys. Rev. D **81**, 084057 (2010) [arXiv:1001.4458 (hep-th)].
- [19] D. Blas, M. Shaposhnikov and D. Zenhausern, arXiv:1104.1392.
- [20] H. Cheng, Phys. Rev. Lett. **61**, 2182 (1988).
- [21] see for example, C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw Hill, 1985.
- [22] D. A. Akyeampong and R. Delbourgo, Nuovo Cim. A **19**, 219 (1974).
- [23] F. Englert, C. Truffin and R. Gastmans, Nucl. Phys. B **117**, 407 (1976).
- [24] M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671**, 162 (2009) [arXiv:0809.3406 [hep-th]].
- [25] M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671**, 187 (2009) [arXiv:0809.3395 [hep-th]].
- [26] P. Jain and S. Mitra, Mod. Phys. Lett. A **24**, 2069 (2009) [arXiv:0902.2525 [hep-ph]].